THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Suggested Solutions to Homework 7

1. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$. Further suppose there exists $x_0 \in \mathbb{R}$ at which f is continuous. Show that there exists a $c \in \mathbb{R}$ such that $f(x) = cx$ for any $x \in \mathbb{R}$.

   Proof. We claim that $c = f(1)$, that is, for $x \in \mathbb{R}$,

   $$f(x) = xf(1).$$

   We will prove this in a sequence of steps:

   Step 1: We will prove that $f(nx) = nf(x)$ for all $n \in \mathbb{Z}$, $x \in \mathbb{R}$.

   First it is easy to note that by linearity,

   $$f(0) = f(0+0) = f(0) + f(0),$$

   which forces to $f(0) = 0$.

   Let $x \in \mathbb{R}$. We have $f(1 \cdot x) = f(x) = 1 \cdot f(x)$. Assume $f(kx) = kf(x)$ for some $k \in \mathbb{N}, k \geq 1$. Then $f((k+1)x) = f(kx + x) = f(kx) + f(x) = kf(x) + f(x) = (k+1)f(x)$. By induction, we have $f(nx) = nf(x)$ for all $n \in \mathbb{N}$.

   More generally, given $n \in \mathbb{Z}$, if $n = 0$ or $n \in \mathbb{N}$, then we are done; otherwise $-n \in \mathbb{N}$, and note that by linearity,

   $$f(nx) + f(-nx) = f(nx + (-nx)) = f(0).$$

   Therefore, $f(nx) = -f(-nx) = -[(-n)f(x)] = nf(x)$. Hence $f(nx) = nf(x)$ for all $n \in \mathbb{Z}$, $x \in \mathbb{R}$.

   Step 2: We show that $f(qx) = qf(x)$ for all $q \in \mathbb{Q}, x \in \mathbb{R}$.

   Write $q = \frac{n}{m}$ in standard form, where $n \in \mathbb{Z}$, $m \in \mathbb{N}$. Then by Step 1,

   $$f(qx) = f(\frac{n}{m} \cdot x) = f(n \cdot \frac{x}{m}) = nf(\frac{x}{m}),$$

   Next, notice that we have $f(x) = f(m \cdot \frac{x}{m}) = mf(\frac{x}{m})$, by linearity. Since we have $m \in \mathbb{N}$, $m \neq 0$. Thus dividing both sides by m, we have $f(\frac{x}{m}) = \frac{1}{m} \cdot f(x)$. By the above, $f(qx) = n\frac{1}{m}f(x) = \frac{n}{m}f(x) = qf(x)$, for any $q \in \mathbb{Q}, x \in \mathbb{R}$.

   Notice that no continuity is needed in Steps 1 and 2.

   Step 3: We claim that $f(x) = xf(1)$ for all $x \in \mathbb{R}$, and this step requires continuity of f. We first claim that continuity at x_0 implies that f is continuous everywhere. Indeed, let $y \in \mathbb{R}$ be given, we will show that f is continuous at y.

Let $\epsilon > 0$. Since f is continuous at x_0, there is $\delta > 0$ such that for all $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$. Now with the same $\delta > 0$, if $|x - y| < \delta$, then $|(x - y + x_0) - x_0| < \delta$. We have, by linearity:

$$|f(x) - f(y)| = |f(x) - f(y) + f(x_0) - f(x_0)| = |f(x - y + x_0) - f(x_0)| < \epsilon,$$

substituting $(x - y + x_0) \mapsto x$. This shows that f is continuous on $\mathbb{R}$.

With the claim above, let $x \in \mathbb{R}$. Then by density of rational numbers, there exists a sequence $(r_n) \subseteq \mathbb{Q}$ such that (r_n) converges to x. By the sequential criterion for continuity, $f(x) = \lim_{n\to\infty} f(r_n)$. But since $r_n \in \mathbb{Q}$, we have $f(r_n) = f(r_n \cdot 1) = r_n f(1)$, by Step 2. Using continuity, we have

$$f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} (r_n f(1)) = (\lim_{n\to\infty} r_n) f(1) = x f(1)$$

Therefore we have shown that $c = f(1) \in \mathbb{R}$ as required. $\square$

2. Let $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) := \begin{cases} 2x, \text{ if } x \in \mathbb{Q} \\ x+3, \text{ if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Find the continuity points of g.

Proof. We claim that g is continuous at $x = 3$ but nowhere else.

Continuity at $x = 3$:

Let $\epsilon > 0$. Note that $g(3) = 2 \times 3 = 6$ since $3 \in \mathbb{Q}$. We choose $\delta := \frac{\epsilon}{2} > 0$. Then for $|x-3| < \delta$,

$$|g(x) - 6| = \begin{cases} 2|x-3| < 2\delta = \epsilon, \text{ if } x \in \mathbb{Q} \\ |x+3-6| = |x-3| < \delta < \epsilon, \text{ if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Therefore g is continuous at $x = 3$.

Discontinuity at $x \neq 3$:

Let $x \neq 3$. Whatever $g(x)$ is, by sequential criterion, it suffices to find two sequences $(x_n), (y_n)$ which converge to x but $g(x_n)$, $g(y_n)$ converge to different limits. To this end, we choose (x_n) be a rational sequence converging to x, and (y_n) be an irrational sequence converging to x, whose existence is guaranteed by density of rational (resp. irrational) numbers in $\mathbb{R}$. Notice that $g(x_n) = 2x_n \to 2x$, $g(y_n) = y_n + 3 \to x + 3$. Since $x \neq 3$, $2x \neq x + 3$, and hence $g(x_n)$, $g(y_n)$ converge to different limits. This shows that g is discontinuous at any $x \neq 3$. □

3. Let $f : A \to \mathbb{R}$, $x_0 \in \mathbb{R}$ a cluster point with respect to A, and suppose that $\lim_{x \to x_0} f(x)$ does not exist in $\mathbb{R}$. Show that there exists $\epsilon > 0$ and two sequences (x_n), $(y_n) \subseteq A \backslash \{x_0\}$ converging to x_0 such that $|f(x_n) - f(y_n)| \geq \epsilon$ for any n.

   If f is bounded, show further that there exist two real numbers $l', l'' \in \mathbb{R}$ and two sequences (x_n') and $(y_n') \subseteq A \backslash \{x_0\}$ converging to x_0 such that $f(x_n')$, $f(y_n')$ converge to l', l'', respectively, but that $l' \neq l''$.

Proof. **First Part**:

We will use a result in Q2, Homework 5, the Cauchy criterion for existence of limits of functions. Since now $\lim_{x \to x_0} f(x)$ does not exist in $\mathbb{R}$, by the negation, there exists $\epsilon > 0$ such that for all $\delta > 0$, there exist $x, x' \in A \backslash \{x_0\}$ with $|x - x_0| < \delta$, $|x' - x_0| < \delta$ such that

$$|f(x) - f(x')| \geq \epsilon.$$

Now for each $n \in \mathbb{N}$, we take $\delta := \frac{1}{n} > 0$, and denote x, x' by x_n, y_n respectively, for each $n \in \mathbb{N}$. In this way we have constructed two sequences (x_n), $(y_n) \subseteq A \backslash \{x_0\}$ converging to x_0 such that $|f(x_n) - f(y_n)| \geq \epsilon$ for any n.

Second Part:

By the first part, we obtain $\epsilon > 0$ and the two sequences $(x_n), (y_n)$ as desired. Since f is bounded, in particular, the sequences $f(x_n), f(y_n)$ are bounded. By Bolzano-Weierstrass theorem, there are real numbers l', l'' and subsequences $f(x_{n_k})$ of $f(x_n)$ and $f(y_{n_k})$ of $f(y_n)$ such that $f(x_{n_k}) \to l$, $f(y_{n_k}) \to l'$. Denote $f(x_{n_k})$ as $f(x_n')$ and $f(y_{n_k})$ as $f(y_n')$.

However, by construction, $|f(x_n') - f(y_n')| \geq \epsilon$ for any n. By the order preserving property, we have

$$|l' - l''| = \lim_{n \to \infty} |f(x_n') - f(y_n')| \geq \epsilon.$$

(Notice that here $\epsilon > 0$ is a fixed constant; not to be confused with the arbitrary $\epsilon > 0$ when we prove the existence of limits)

In particular, we have shown that $l' \neq l''$. □

4. Consider real numbers $a < b < c$. Let $f : (a, b] \to \mathbb{R}$, $g : [b, c) \to \mathbb{R}$ be continuous at b, and suppose that $f(b) = g(b)$. Let $h : (a, c) \to \mathbb{R}$ be defined by:

$$h(x) := \begin{cases} f(x), & \text{if } x \in (a, b] \\ g(x), & \text{if } x \in [b, c) \end{cases}$$

Show that

(a)h is continuous at b.

(b)If f,g are uniformly continuous then so is h.

Proof. (a) Note that the condition that $f(b) = g(b)$ ensures that $h(b)$ is well-defined. Let $\epsilon > 0$. Since f is continuous at b, there is $\delta_1 > 0$ such that for each $b - \delta_1 < x \le b$, $x > a$,

$$|f(x) - f(b)| < \epsilon.$$

Similarly, since g is continuous at b, there is $\delta_2 > 0$ such that for each $b \le x < x + \delta_2$, $x < c$,

$$|g(x) - g(b)| < \epsilon.$$

Then take $\delta := \min\{\delta_1, \delta_2\} > 0$. For $b - \delta < x < b + \delta$, $a < x < c$, we have:

$$|h(x) - h(b)| = \begin{cases} |f(x) - f(b)| < \epsilon, & \text{if } a < x \le b \\ |g(x) - g(b)| < \epsilon, & \text{if } b \le x < c \end{cases}$$

Hence g is continuous at b.

(b) Let $\epsilon > 0$. Since f is uniformly continuous, there is $\delta_3 > 0$ such that for each $|x - y| < \delta_3, a < x \le b, a < y \le b$,

$$|f(x) - f(y)| < \frac{\epsilon}{2}.$$

Similarly, since g is uniformly continuous, there is $\delta_4 > 0$ such that for each $|x - y| < \delta_4, b \le x < c, b \le y < c$,

$$|g(x) - g(y)| < \frac{\epsilon}{2}.$$

Then take $\delta' := \min\{\delta_3, \delta_4\} > 0$. For $|x - y| < \delta', a < x < c, a < y < c$, we have:

$$|h(x) - h(y)| = \begin{cases} |f(x) - f(y)| < \frac{\epsilon}{2} < \epsilon, & \text{if } a < x \le b, a < y \le b \\ |g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon, & \text{if } b \le x < c, b \le y < c \\ |f(x) - g(y)| \le |f(x) - f(b)| + |g(y) - g(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, & \\ \text{if } a < x \le b, b \le y < c & \\ |g(x) - f(y)| \le |g(x) - g(b)| + |f(y) - f(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, & \\ \text{if } b \le x < c, a < y \le b & \end{cases}$$

Hence h is uniformly continuous on (a, c). □